A Linear Programming View on Plastic Structural Analysis

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Introduction

Simple Example i.

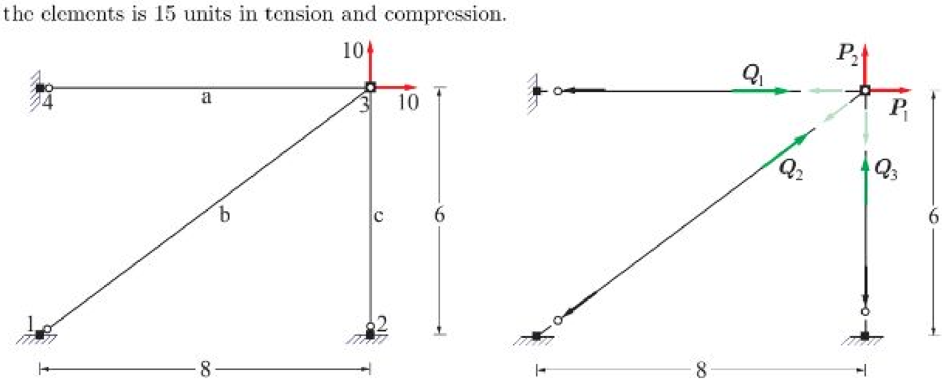
We first begin with a simple example as stated below (*CE220 Structural Analysis Textbook, Prof. Filippou*):

Example 12.1 Plane Truss

The plane truss in Fig. 12.2(a) is subjected to nodal forces in the X- and Y-direction at the only free node. The given applied load distribution is represented by the reference load vector $\boldsymbol{P}_{ref}$

$$\boldsymbol{P}_{ref} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

The plane truss has 3 elements, so that the number of basic element forces nq is 3. The axial capacity of the elements is 15 units in tension and compression.



(a) Geometry and loading

(b) Free dof equation and basic force numbering

Fig. 12.2: Geometry, loading, equilibrium equation and basic force numbering for truss model

Fig. 1

The system of equations is:

$$\begin{aligned} \lambda\,(10) &= \boldsymbol{Q}_1 + 0.8\boldsymbol{Q}_2 \\ \lambda\,(10) &= \qquad 0.6\boldsymbol{Q}_2 + \boldsymbol{Q}_3 \end{aligned}$$

Fig. 2

Formulation

The original linear programming formulation is:

(LP1) $\quad$ *maximize* λ

$$
\begin{aligned}
st.\quad -(10)\lambda + Q_1 + 0.8Q_2 &= 0 \\
-(10)\lambda + 0.6Q_2 + Q_3 &= 0 \\
C_1^{-} \le Q_1 \le C_2^{+} & \\
C_2^{-} \le Q_2 \le C_3^{+} & \\
C_3^{-} \le Q_3 \le C_3^{+} & \\
Q_1,\ Q_{2,}\ Q_3 \ \textit{sign unrestricted} & \\
\lambda \ge 0 &
\end{aligned}
$$

where C_i^{-} is the negative capacity and C_i^{+} is the positive capacity of Q_i, $\forall i \in \{1,2,3\}$.

Preprocessing

i. ***Bound Tightening:*** Note that in the original problem formulation, all coefficients of $Q_i,\ \forall i \in \{1,2,3\}$ are negative (all positive in the structural equilibrium system of equations in Fig. 2). Because $\lambda \ge 0$, it is intuitive to maximize $Q_i,\ \forall i \in \{1,2,3\}$ in order to maximize λ. An equally valid formulation with a tighter bound can be found:

(LP2) $\quad$ *maximize* λ

$$
\begin{aligned}
st.\quad -(10)\lambda + Q_1 + 0.8Q_2 &= 0 \\
-(10)\lambda + 0.6Q_2 + Q_3 &= 0 \\
Q_1 \le C_1^{+} & \\
Q_2 \le C_2^{+} & \\
Q_3 \le C_3^{+} & \\
Q_1,\ Q_{2,}\ Q_3 \ \ge 0 & \\
\lambda \ge 0 &
\end{aligned}
$$

This observation is crucial for saving computational complexity in our algorithm later on.

ii. ***Choose redundant variables:*** For a statically indeterminate structure (in this case, NOS = 1), we can separate the solution into particular and homogeneous parts (refer to *CE220 Structural Analysis, Prof. Filippou*). This involves arbitrarily setting Q_1 and Q_3 as basic variables and applying Gaussian Elimination to pivot on Q_1 and Q_2. The solution is written as:

$$
\boldsymbol{Q} = \lambda \boldsymbol{Q}_{pr} + \bar{\mathbf{B}}_x \boldsymbol{Q}_x = \lambda \begin{pmatrix} 10 \\ 0 \\ 10 \end{pmatrix} + \begin{pmatrix} -0.8 \\ 1 \\ -0.6 \end{pmatrix} \boldsymbol{Q}_2 \qquad (12.18)
$$

Fig. 3

Write out the above equation entry by entry:

$$Q_1 = \lambda(10) - (0.8)Q_2$$
$$Q_2 = \lambda(0) + Q_2$$
$$Q_3 = \lambda(10) - (0.6)Q_2$$

Substitute the above equations into the capacity inequalities:

$$Q_1 = \lambda(10) - (0.8)Q_2 \le C_1^{+}$$
$$Q_2 = \lambda(0) + Q_2 \le C_2^{+}$$
$$Q_3 = \lambda(10) - (0.6)Q_2 \le C_3^{+}$$

The problem formulation now becomes:

(LP3) $\quad$ *maximize* λ

$$\begin{aligned} st.\ & \lambda(10) - (0.8)Q_2 \le C_1^{+} \\ & \lambda(0) + Q_2 \le C_2^{+} \\ & \lambda(10) - (0.6)Q_2 \le C_3^{+} \\ & Q_1,\ Q_{2,}\ Q_3 \ge 0 \\ & \lambda \ge 0 \end{aligned}$$

This step is crucial for reducing the computation complexity of the algorithm.

iii. ***Preprocess Inequalities:*** In order to turn the capacity inequalities to equalities, we introduce positive slack variables to the formulation:

$$\begin{aligned} & \lambda(10) - (0.8)Q_2 + S_1 = C_1^{+} \\ & \lambda(0) + Q_2 + S_2 = C_2^{+} \\ & \lambda(10) - (0.6)Q_2 + S_3 = C_3^{+} \\ & S_1,\ S_2,\ S_3 \ge 0 \end{aligned}$$

(LP3) now becomes:

(LP4) $\quad$ *maximize* λ

$$\begin{aligned} st.\ & \lambda(10) - (0.8)Q_2 + S_1 = C_1^{+} \\ & \lambda(0) + Q_2 + S_2 = C_2^{+} \\ & \lambda(10) - (0.6)Q_2 + S_3 = C_3^{+} \\ & Q_1,\ Q_{2,}\ Q_3,\ S_1,\ S_2,\ S_3 \ge 0 \\ & \lambda \ge 0 \end{aligned}$$

Tabulating the variables' coefficients:

Row No.	λ	Q_2	S_1	S_2	S_3	RHS

(1)	10	-0.8	1	0	0	C_1^+
(2)	0	1	0	1	0	C_2^+
(3)	10	-0.6	0	0	1	C_3^+

Where in this specific problem, $C_1^+ = 15,\ C_2^+ = 15,\ C_3^+ = 15$.

Iterations

Initialization

The tableau matrix formed at the end of the preprocessing steps has more variables than rows. It is an indeterminate system. We have to split the variables into two sets: basic variables and nonbasic variables:

i. ***Basic Variables*** are variables that have non-zero values. The number of basic variables is equal to the number of rows of the system (in this case, 3). There is only one basic variable per row, and its value is strictly equal to the right hand side of that row divided by the basic variable's coefficient.

ii. ***Nonbasic Variables*** are variables that are strictly zero.

The ***goal*** of the algorithm is to find the optimal way to split the variables into basic and nonbasic sets, while satisfying the equality relationships. Optimal implies a solution that maximizes λ.

NOTE: By Theorem, optimum occurs at extreme points. Nonbasic (redundant) variables in the tableau has to be zero for a possible optimum to occur. (Need more explanation here)

We add a new column, "Basic Var" to keep track of each row's basic variable at each iteration step.

Iter 0 Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	10	-0.8	1	0	0	$C_1^+ = 15$	S_1
(2)	0	1	0	1	0	$C_2^+ = 15$	S_2
(3)	10	-0.6	0	0	1	$C_3^+ = 15$	S_3

For better demonstration, perform row operations so that λ has coefficients = 1 in Row No (1) and (3):

Iter 0' Tableau--reduce λ's coefficients:

Row No.(1)/10 -> Row No. (1);

Row No.(3)/10 -> Row No. (3):

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	-0.08	0.1	0	0	1.5	S_1
(2)	0	1	0	1	0	15	S_2
(3)	1	-0.06	0	0	0.1	1.5	S_3

λ is a nonbasic variable, so $\lambda = 0$. There is no loading to the structure at this point (Iter 0).

Iteration 1

Choose leaving basic variable

We want to increase λ. In order for λ to become basic, it has to replace a currently nonbasic variable. What is the maximum allowable value for λ? We take a look at each row:

Row No. (1):
$(1)\lambda - 0.08Q_2 + S_1 = 1.5$. Q_2 is nonbasic, so $Q_2 = 0$. In order to increase λ, S_1, which is the basic variable, should decrease in order to "make room" for the increment of λ. Ignore Q_2 and rearrange to get: $(1)\lambda = 1.5 - S_1$. By definition, $S_1 \geq 0$, so $1.5 - S_1 \leq 1.5$, which means $(1)\lambda \leq 1.5$. λ can increase to a maximum value of 1.5 while still satisfying the feasible region (ie. still making sure $S_1 \geq 0$).

Row No. (2):
$(0)\lambda + Q_2 + S_2 = 15$. Q_2 is nonbasic, so $Q_2 = 0$. With the similar reasoning presented for the case of Row No. (1), $(0)\lambda \leq 15$. All values of λ satisfy this inequality, so there is no restriction on λ by Row No. (2).

Row No. (3):
$(1)\lambda - 0.06Q_2 + S_3 = 1.5$. Similarly, $Q_2 = 0$, $S_3 \geq 0$. $\lambda \leq 1.5$

In order to satisfy the feasible region, all the restrictions imposed by Row No. (1), (2), and (3) should apply: {$\lambda \leq 1.5$; $(0)\lambda \leq 20$; $\lambda \leq 1.5$}. The tightest bound is $\lambda \leq 1.5$, as imposed by Row No. (1) and (3). λ can "safely" increase to 1.5 without violating the problem boundary. This drives S_1 or S_3 to 0, making S_1 or S_3 nonbasic. However, at each iteration, only one nonbasic variable becomes basic (in this case, λ becomes basic), replacing one already basic variable (replacing either S_1 or S_3). We arbitrary break the tie and let λ enter Row. No. (1), replacing S_1:

Make λ basi in Row No.(1):

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var

(1)	1	-0.08	0.1	0	0	1.5	λ
(2)	0	1	0	1	0	15	S_2
(3)	1	-0.06	0	0	0.1	1.5	S_3

We call the above procedure of determining a leaving basic variable the "*ratio test*", which is: the leaving abasic variable is one whose row's RHS divided by the coefficient of the entering variable is the lowest ***non-negative*** number.

Now because λ is no longer zero, the RHS of other rows (where λ has a non-zero coefficient) might be subject to change (why?). Applying Gaussian Elimination and pivot on λ, so that λ has *zero coefficient* in Row No. (2) and (3), so that ***we don't have to account for* λ *on the right hand sides of Row No. (2) and (3):***

Iter 1 Tableau

Row No.(3) - Row No.(1) -> Row No.(3):

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	-0.08	0.1	0	0	1.5	λ
(2)	0	1	0	1	0	15	S_2
(3)	0	0.02	-0.1	0	0.1	0	S_3

Or, if λ enters Row No. (3):

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	0	-0.2	1	0	-1	0	S_1
(2)	0	1	0	1	0	15	S_2
(3)	1	-0.06	0	0	0.1	1.5	λ

Now Row No. (1), (2), and (3) have finished their update at iter 1.

Structurally speaking: The above steps mean that by first setting the redundant force Q_2 to zero, we find a lower bound of λ based on the geometry of Q_1 and Q_2. This is the maximum allowable load factor if we add a hinge on Q_2. Q_2 is not "helping" neither Q_1 nor Q_2.

Iteration 2

Choose entering variable

Now λ is given a lower bound ($\lambda = 1.5$), although it is not optimal.
This lower bound indicates the scenario where Q_2 is zero (we can interpret it as adding a hinge to Q_2 so that we are only looking at the *primary* structure). How does Q_2 actually influence λ? We look at Row No. (1) (Iter 1 Tableau replicated below):

Iter 1 Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	-0.08	0.1	0	0	1.5	λ
(2)	0	1	0	1	0	15	S_2
(3)	0	0.02	-0.1	0	0.1	0	S_3

$\lambda - 0.08Q_2 + 0.1S_1 = 1.5$; $\lambda + 0.1S_1 = 1.5 + 0.08Q_2$; S_1 is nonbasic, so substitute $S_1 = 0$:
$\lambda = 1.5 + 0.08Q_2$. There is a linear relationship between Q_2 and λ, with $slope = 0.08$ and $\lambda_{current} = 1.5$.
Increment of Q_2 by $1\ unit$ increases λ by $0.08\ unit$. Therefore we conclude that *the rate of change of λ with respect to a variable is equal to the negated coefficient of that variable in the row where λ is a basic variable (with coefficient = 1). (theorem 1.1)*
Similarly, in Row No. (1), because S_1's coefficient is 0.1, increasing S_1 by $1\ unit$ will increase λ by -0.1 (negated coefficient).

Thus between Q_2 and S_2, we choose to enter Q_2 to the basic variables set.

Choose leaving variable

Continue entering Q_2 as the new basic variable. Which basic variable should Q_2 replace (ie, what is the leaving variable)? We conduct the ratio test:
Row No. (1): $-0.08Q_2 \leq 1.5$; $Q_2 \geq -18.85$ (redundant, as Q_2 is defined as nonnegative)
Row No. (2): $Q_2 \leq 15$
Row No. (3): $0.02Q_2 \leq 0$; $Q_2 \leq 0$
The ratio test dictates that Q_2 should enter Row No. (3), because Row No.(3) provides the tightest bound (most limiting bound) in combination with $Q_2 \geq 0$ (definition).

Now we ***pivot*** on Q_2 in Row No. (3)--make Q_2's coefficient in Row No. (3) equal to 1 and row reduce other rows so that Q_2 has zero coefficients in other rows (so we don't have to worry about the effect of increased Q_2 value in other rows). However, for computational convenience make sure all the other basic variables should still have coefficients=1 in their respective rows:

Iter 2 Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	0	-0.3	0	0.4	1.5	λ
(2)	0	0	5	1	-5	15	S_2
(3)	0	1	-5	0	5	0	$S_3 \text{->} Q_2$

Iteration 2 is now compete.

Iteration 3

Choose the next entering variable

Following the rule that *the rate of change of λ with respect to a variable is equal to the negated coefficient of that variable in the row where λ is a basic variable. (theorem 1.1):*
we choose S_1 as the next entering variable

Choose leaving variable:

S_1 enters basic set. What is replaced by S_1?
Ratio Test:

Row No. (1): $-0.3S_1 \le 1.5$; $S_1 \ge -5$ (redundant, because S_1 is defined as nonnegative)
Row No. (2): $5S_1 \le 15$; $S_1 \le 3$ (binding)
Row No. (3): $-5S_1 \le 0$; $S_1 \ge 0$ (redundant)

S_1 thus enters Row No. (2), replacing S_2. Pivot on S_1 in Row No.(2) to get:
Iter 3 Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	0	0	0.06	0.1	2.4	λ
(2)	0	0	1	0.2	-1	3	$S_2 \text{->} S_1$
(3)	0	1	0	1	0	15	Q_2

Iteration 3 is complete

Iteration 4

Choose entering variable:

Take a look at Row No. (1) of Iter 3 Tableau: only S_2 *and* S_3 are nonbasic variables with nonzero coefficients.

if enter S_2:

$\lambda + 0.06S_2 = 2.4$; $\lambda = 2.4 - 0.06S_2$. Increasing S_2 will NOT help increase λ.

if enter S_3:

$\lambda + 0.1S_3 = 2.4$; $\lambda = 2.4 - 0.1S_3$. Increasing S_3 will NOT help increase λ.

if enter $Q_1, Q_{2,} Q_{3,}$ *or* S_1 (nonbasic variables with zero coefficients):

$\lambda + (0)\{Q_1, Q_{2,} Q_{3,} \text{ or } S_1\} = 2.4$; $\lambda = 2.4 - (0)\{Q_1, Q_{2,} Q_{3,} \text{ or } S_1\}$. Increasing $Q_1, Q_{2,} Q_{3,}$ *or* S_1 will NOT help increase λ.

We are out of ways to improve/increase λ, so the optimal $\lambda = 2.4$. The algorithm terminates.
The algorithm terminates when all the nonbasic variables' coefficients in the row where λ is a basic variable are non-negative. The algorithm terminates when there is no way to improve λ.

Algorithm Summary:

Preprocessing:

i. Bound Tightening: in the structural equilibrium equations, if the all coefficients of Q_i are:
positive: to maximize λ (nonnegative), all Q_i should be positive or zero;
negative: to maximize λ (nonnegative), all Q_i should be negative or zero.

Example:

(LP2) *maximize* λ

$$
\begin{aligned}
st.\quad & -(10)\lambda + Q_1 + 0.8Q_2 = 0 \\
& -(10)\lambda + 0.6Q_2 + Q_3 = 0 \\
& Q_1 \leq C_1^{\,+} \\
& Q_2 \leq C_2^{\,+} \\
& Q_3 \leq C_3^{\,+} \\
& Q_1, Q_{2,} Q_3 \geq 0 \text{ (All Q's are positive or zero)} \\
& \lambda \geq 0
\end{aligned}
$$

ii. Choose redundant variables and write all Q_i in terms of Q_x (redundant variables):

$$\boldsymbol{Q} = \lambda \boldsymbol{Q}_{pr} + \bar{\mathbf{B}}_x \boldsymbol{Q}_x = \lambda \begin{pmatrix} 10 \\ 0 \\ 10 \end{pmatrix} + \begin{pmatrix} -0.8 \\ 1 \\ -0.6 \end{pmatrix} \boldsymbol{Q}_2 \tag{12.18}$$

Example:

$Q_1 = \lambda(10) - (0.8)Q_2$
$Q_2 = \lambda(0) + Q_2$
$Q_3 = \lambda(10) - (0.6)Q_2$

iii. Express each Q_i 's strength boundary in terms of redundant variables:

Example:

$Q_1 = \lambda(10) - (0.8)Q_2 \le C_1^+$
$Q_2 = \lambda(0) + Q_2 \le C_2^+$
$Q_3 = \lambda(10) - (0.6)Q_2 \le C_3^+$
$Q_1, Q_2, Q_3 \ge 0$

iv. Add nonnegative slack variables (unused potential strength):

Example:

(LP4) *maximize* λ

$$\begin{aligned} st.\ & \lambda(10) - (0.8)Q_2 + S_1 = C_1^+ \\ & \lambda(0) + Q_2 + S_2 = C_2^+ \\ & \lambda(10) - (0.6)Q_2 + S_3 = C_3^+ \\ & Q_1, Q_2, Q_3, S_1, S_2, S_3 \ge 0 \\ & \lambda \ge 0 \end{aligned}$$

Tableau Initialization:

Initialize to tableau:

Iter 0 Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	10	-0.8	1	0	0	$C_1^+ = 15$	S_1
(2)	0	1	0	1	0	$C_2^+ = 15$	S_2
(3)	10	-0.6	0	0	1	$C_3^+ = 15$	S_3

Iteration Processes:

Iteration 1:

The first iteration enters λ to the basic variables set. Perform ratio test and enter λ into the row with the lowest ratio (RHS/nonnegative coefficient of λ). Arbitrarily break tie. Then perform row operations to pivot on λ in the row it just entered.

Iter 1 Tableau

Row No.(3) - Row No.(1) -> Row No.(3):

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	-0.08	0.1	0	0	1.5	λ
(2)	0	1	0	1	0	15	S_2
(3)	0	0.02	-0.1	0	0.1	0	S_3

Iteration 2 and beyond:

For each iteration:

- *Choose entering variable:*

Pick a nonbasic variable that has the ***most negative*** coefficient in the row where λ is a basic variable.

- Choose leaving variable:

Enter that variable to the row where its lowest ratio test (RHS/nonnegative coefficient of entering variable), replacing the variable previously in that row (called leaving variable).

Iter 2 Tableau (Q_2 has the most negative coefficient in Row No.(1) in Iter 1 Tableau. It enters Row No.(3) where it has the lowest ratio test)

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	0	-0.3	0	0.4	1.5	λ
(2)	0	0	5	1	-5	15	S_2
(3)	0	1	-5	0	5	0	$S_3 \to Q_2$

Repeat above loop and update tableau

Termination Condition:

The algorithm terminates when:

in the row where λ is a basic variable, all nonbasic variables have non-negative coefficients.

The optimal λ value is the RHS of that row.

Termination Tableau

Row.No.	λ	Q_2	S_1	S_2	S_3	RHS	Basic Var
(1)	1	0	0	0.06	0.1	2.4	λ
(2)	0	0	1	0.2	-1	3	S_1
(3)	0	1	0	1	0	15	Q_2